Problem 1-5

Before Planck's theoretical work on blackbody radiation, Wien showed empirically that (Equation 1.4)

$$\lambda_{\text{max}} T = 2.90 \times 10^{-3}\ \text{m} \cdot \text{K}$$

where λ_{max} is the wavelength at which the blackbody spectrum has its maximum value at a temperature T. This expression is called the Wien displacement law; derive it from Planck's theoretical expression for the blackbody distribution by differentiating Equation 1.3 with respect to λ. *Hint*: Set $hc/\lambda_{\text{max}} k_B T = x$ and derive the intermediate result $e^{-x} + (x/5) = 1$. This problem cannot be solved for x analytically but must be solved numerically. Solve it by iteration on a hand calculator, and show that $x = 4.965$ is the solution.

Solution

The Planck distribution law for blackbody radiation is given by

$$\rho_\nu(T)\, d\nu = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/k_B T} - 1}\, d\nu.$$

Use the chain rule to write this in terms of the wavelength λ.

$$\rho_\nu(T) \left(\frac{d\nu}{d\lambda}\right) d\lambda = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/k_B T} - 1} \left(\frac{d\nu}{d\lambda}\right) d\lambda \tag{1}$$

The relationship between frequency and wavelength for blackbody radiation is

$$\lambda\nu = c \quad \Rightarrow \quad \begin{cases} \nu = \dfrac{c}{\lambda} \\ \dfrac{d\nu}{d\lambda} = -\dfrac{c}{\lambda^2} \end{cases}.$$

As a result, equation (1) becomes

$$\begin{aligned} [-\rho_\lambda(T)]\, d\lambda &= \frac{8\pi h}{c^3} \frac{\left(\frac{c}{\lambda}\right)^3}{\exp\left(\frac{hc}{k_B T\lambda}\right) - 1} \left(-\frac{c}{\lambda^2}\right) d\lambda \\ &= -\frac{8\pi hc}{\lambda^5} \frac{d\lambda}{\exp\left(\frac{hc}{k_B T\lambda}\right) - 1}. \end{aligned}$$

Therefore, the Planck distribution in terms of wavelength is

$$\rho_\lambda(T)\, d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{\exp\left(\frac{hc}{k_B T\lambda}\right) - 1}.$$

In order to find the wavelength at which the spectrum has its maximum value, differentiate $\rho_\lambda(T)$ with respect to λ and set it equal to zero.

$$\frac{d}{d\lambda}\rho_\lambda(T) = 0$$

Substitute the distribution and evaluate the derivative.

$$\begin{aligned}
0 &= \frac{d}{d\lambda}\left[\frac{8\pi hc}{\lambda^5}\frac{1}{\exp\left(\frac{hc}{k_BT\lambda}\right)-1}\right] \\
&= 8\pi hc\frac{d}{d\lambda}\left[\frac{1}{\lambda^5}\frac{1}{\exp\left(\frac{hc}{k_BT\lambda}\right)-1}\right] \\
&= 8\pi hc\left\{-\frac{5}{\lambda^6}\frac{1}{\exp\left(\frac{hc}{k_BT\lambda}\right)-1}+\frac{1}{\lambda^5}\frac{-\exp\left(\frac{hc}{k_BT\lambda}\right)\left(-\frac{hc}{k_BT\lambda^2}\right)}{\left[\exp\left(\frac{hc}{k_BT\lambda}\right)-1\right]^2}\right\} \\
&= 8\pi hc\left\{-\frac{5}{\lambda^6}\frac{\exp\left(\frac{hc}{k_BT\lambda}\right)-1}{\left[\exp\left(\frac{hc}{k_BT\lambda}\right)-1\right]^2}+\frac{\lambda}{\lambda^6}\frac{-\exp\left(\frac{hc}{k_BT\lambda}\right)\left(-\frac{hc}{k_BT\lambda^2}\right)}{\left[\exp\left(\frac{hc}{k_BT\lambda}\right)-1\right]^2}\right\} \\
&= 8\pi hc\left\{\frac{-5\exp\left(\frac{hc}{k_BT\lambda}\right)+5}{\lambda^6\left[\exp\left(\frac{hc}{k_BT\lambda}\right)-1\right]^2}+\frac{\frac{hc}{k_BT\lambda}\exp\left(\frac{hc}{k_BT\lambda}\right)}{\lambda^6\left[\exp\left(\frac{hc}{k_BT\lambda}\right)-1\right]^2}\right\} \\
&= 8\pi hc\frac{\left(-5+\frac{hc}{k_BT\lambda}\right)\exp\left(\frac{hc}{k_BT\lambda}\right)+5}{\lambda^6\left[\exp\left(\frac{hc}{k_BT\lambda}\right)-1\right]^2}
\end{aligned}$$

Divide both sides by $8\pi hc$.

$$0 = \frac{\left(-5+\frac{hc}{k_BT\lambda}\right)\exp\left(\frac{hc}{k_BT\lambda}\right)+5}{\lambda^6\left[\exp\left(\frac{hc}{k_BT\lambda}\right)-1\right]^2}$$

Multiply both sides by $\lambda^6\left[\exp\left(\frac{hc}{k_BT\lambda}\right)-1\right]^2$.

$$0 = \left(-5+\frac{hc}{k_BT\lambda}\right)\exp\left(\frac{hc}{k_BT\lambda}\right)+5$$

Let

$$x = \frac{hc}{k_BT\lambda}. \tag{2}$$

Then

$$0 = (-5+x)e^x+5.$$

Solving this equation numerically yields $x \approx 4.96511$. Equation (2) then becomes

$$4.96511 \approx \frac{hc}{k_BT\lambda}.$$

Therefore, setting $\lambda = \lambda_{\text{max}}$ and solving for $\lambda_{\text{max}}T$, we get the Wien displacement law.

$$\lambda_{\text{max}}T \approx \frac{hc}{(4.96511)k_B} \approx \frac{(6.626\times10^{-34}\ \cancel{\text{J}}\cdot\cancel{\text{s}})\left(299\,792\,458\ \frac{\text{m}}{\cancel{\text{s}}}\right)}{4.96511(1.38\times10^{-23}\ \cancel{\text{J}}\cdot\text{K}^{-1})} \approx 2.90\times10^{-3}\ \text{m}\cdot\text{K}$$